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Mazur–Orlicz Theorem in Lifting Theory

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We apply the Mazur–Orlicz theorem for the reconstruction of linear liftings from the classes of subadditive resp. superadditive liftings. We check the maximal and minimal elements of these classes and relate them to (linear) liftings.

INTRODUCTION

The problem of projecting liftings from product probability spaces into the factors turned up different situations (see, e.g., [8, 9]). It is know that this process causes a considerable loss in structure, (linear) liftings are transformed into subadditive resp. superadditive liftings. The paper deals with the problem of how to reconstruct the structure as far as possible. We show that by means of the Mazur–Orlicz theorem we can reconstruct linear liftings (see 2.9), but (multiplicative) liftings cannot be reconstructed generally (see 2.2).

Though von Neumann never defined subadditive resp. superadditive liftings explicitly, they occur naturally in the generalization of a process given by him, but with additional compatibilities under lattice operations.

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We study the corresponding classes in Section 2. As a consequence of the Mazur–Orlicz theorem we find that the class of all linear liftings coincides with the system of all maximal superadditive liftings (see Theorem 2.20) and give a corresponding result for liftings instead of linear liftings (see 2.24) provided the basic measure spaces are complete.

1. PRELIMINARIES

For a measure space (Ω, Σ, μ) a set $N \in \Sigma$ with $\mu(N) = 0$ is called a *μ-null set*. The family of all μ-null sets is denoted by Σ_0. For $A, B \in \Sigma$ we write $A = B$ a.e. (μ), if and only if $A \triangle B$, the symmetric difference of A and B, is a μ-null set. For $A \in \Sigma$ we denote by $\hat{A}$ the class of all $B \in \Sigma$ such that $A \triangle B \in \Sigma_0$. The (Carathéodory) completion of (Ω, Σ, μ) will be denoted by $(\Omega, \hat{\Sigma}, \hat{\mu})$. The system of all real resp. rational numbers will be denoted by $\mathbf{R}$ resp. $\mathbf{Q}$ and $\overline{\mathbf{R}}$ denotes the set $\mathbf{R} \cup \{-\infty, +\infty\}$. We write $\mathscr{L}^\infty(\mu)$ for the space of all Σ-measurable functions from Ω into $\mathbf{R}$ with $\sup_{\omega \in \Omega} |f(\omega)| < \infty$, and $\|f\|_\infty$ for the ess sup $|f|$, if $f \in \mathscr{L}^\infty(\mu)$.

$\overline{\mathscr{L}}^0(\mu)$ denotes the space of all measurable functions from Ω into $\overline{\mathbf{R}}$. $\mathscr{L}^0(\mu)$ denotes the space of all measurable functions from Ω into $\mathbf{R}$.

For $f, g \in \mathscr{L}^\infty(\mu)$ we write $f = g$ a.e. (μ), if and only if $\{\omega \in \Omega : f(\omega) \neq g(\omega)\} \in \Sigma_0$. For $f \in \mathscr{L}^\infty(\mu)$ we denote by $\hat{f}$ the class of all $g \in \mathscr{L}^\infty(\mu)$ such that $f = g$ a.e. (μ). The same symbols will be used for elements of $\mathscr{L}^0(\mu)$ (or $\overline{\mathscr{L}}^0(\mu)$) instead of $\mathscr{L}^\infty(\mu)$. For two functions f, g from Ω into $\overline{\mathbf{R}}$ we write $f \vee g := \sup(f, g)$ and $f \wedge g := \inf(f, g)$. We write $\Lambda^\infty(\mu)$ for the system of all (multiplicative) liftings of $\mathscr{L}^\infty(\mu)$ in the sense of [6, Chapter III, Section 1, Definition 2] and $\Lambda(\mu)$ for the system of all (Boolean) liftings for sets in Σ, where we do not distinguish between the lifting ρ for Σ and the lifting ρ^∞ on $\mathscr{L}^\infty(\mu)$ related in a biunique way to each others by means of the equation $\rho^\infty(\chi_A) = \chi_{\rho(A)}$ for all $A \in \Sigma$ (see [6, p. 35]) if it is clear from the context what is meant. $\mathscr{G}(\mu)$ resp. $\vartheta(\mu)$ resp. $Y(\mu)$ is the system of all linear liftings resp. lower densities resp. upper densities for μ in the sense of [6, Chapter III, Section 1, Definition 1. resp. Definition 4].

All unexplained notions concerning liftings can be found in A. Ionescu and C. Ionescu [6].

2. PROJECTIVE LIFTINGS IN PRODUCTS

Let us be given two complete probability spaces (Ω, Σ, μ) and (Θ, T, ν). For $\rho \in \mathscr{G}(\mu \hat{\otimes} \nu)$ define the *outer* resp. *inner projective lifting* of ρ under the canonical map $p_1\colon \Omega \times \Theta \to \Omega$, written ρ^* resp. ρ_*, by means of

$$\rho^*(f)(\omega) := \int^* [\rho(f \circ p_1)]_\omega \, d\nu$$

and

$$\rho_*(f)(\omega) := \int_* [\rho(f \circ p_1)]_\omega \, d\nu$$

for any $\omega \in \Omega$ and $f \in \mathscr{L}^\infty(\mu)$. Compare, e.g., [7, III, 3] for the definition and the elementary properties of the upper resp. lower integral. We apply [7, III, 3] with $\mathscr{E} := \mathscr{L}^1(\mu)$ and $m(f) := \int f \, d\mu$ for $f \in \mathscr{L}^\infty(\mu)$.

PROPOSITION 2.1. *For complete probability spaces (Ω, Σ, μ) and (Θ, T, ν) the projective liftings ρ^* and ρ_* map $\mathscr{L}^\infty(\mu)$ into itself in such a way that for any $f, g \in \mathscr{L}^\infty(\mu)$ we have*

(l1) $\rho^*(f) = \rho_*(f) = f$ *a.e.* (μ).

(l2) $\rho^*(f) = \rho^*(g)$ *and* $\rho_*(f) = \rho_*(g)$, *if* $f = g$ *a.e.* (μ).

(l3) $\rho^*(\alpha) = \rho_*(\alpha) = \alpha$ *for any constant* α.

(l4) $0 \le f$ *implies* $0 \le \rho^*(f)$ *and* $0 \le \rho_*(f)$.

(l5) $f \le g$ *implies* $\rho^*(f) \le \rho^*(g)$ *and* $\rho_*(f) \le \rho_*(g)$.

(l6) $\rho^*(\alpha f) = \alpha \rho^*(f)$ *and* $\rho_*(\alpha f) = \alpha \rho_*(f)$ *for any* $\alpha \ge 0$.

(l7) $\rho^*(f + g) \le \rho^*(f) + \rho^*(g)$.

(l8) $\rho_*(f + g) \ge \rho_*(f) + \rho_*(g)$.

(l9) $\rho_* \le \rho^*$ *on* $\mathscr{L}^\infty(\mu)$.

(l10) $\rho^*(f) = \rho_*(f) = \int [\rho(f \circ p_1)]_\omega \, d\nu$ *if* $[\rho(f \circ p_1)]_\omega \in \mathscr{L}^\infty(\nu)$ *for all* $\omega \in \Omega$.

(l11) $\rho^*(f + a) = \rho^*(f) + a$ *and* $\rho_*(f + a) = \rho_*(f) + a$ *for all* $a \in \mathbf{R}$.

Proof. Ad (l1) note that for $f \in \mathscr{L}^\infty(\nu)$ there exists a set $N_f \in \Sigma_0$ such that for any $\omega \in \Omega \setminus N_f$ we have

$$[\rho(f \circ p_1)]_\omega \in \mathscr{L}^\infty(\nu).$$

So for any $\omega \in \Omega \setminus N_f$ we get

$$\int^* [\rho(f \circ p_1)]_\omega \, d\nu = \int [\rho(f \circ p_1)]_\omega \, d\nu = \int [f \circ p_1]_\omega \, d\nu = f(\omega);$$

hence $\rho^*(f) = f$ i.e. (μ) and in the same way $\rho_*(f) = f$ a.e. (μ).

The rest is straightforward to verify starting from the properties of the outer resp. inner integral. ∎

Usually neither ρ_* nor ρ^* will be linear or multiplicative, but in view of (l9) the question turns up whether there exists an $\upsilon \in \Lambda(\mu)$ with

$$\rho_* \le \upsilon \le \rho^*. \tag{1}$$

But this is not true as witnessed by the next example.

EXAMPLE 2.2. Let $(\Omega, \Sigma, \mu) = (\Theta, T, \nu)$ be the Lebesgue measure space on $[0,1]$ and put

$$A := \{(\omega, \theta) \in \Omega \times \Theta : 0 \le \omega < 1/2, 0 \le \theta \le 1\}.$$

Define a lifting ρ on the σ-algebra generated by the set A and the null sets in $\widehat{\Sigma^2}$ by means of

$$\rho(A) := A \cup (\{1/2\} \times [0, 1/2])$$

and then extend it to $\widehat{\Sigma^2}$ by means of [6, Chapter IV, Section 2, Theorem 4].

Denote again by ρ the corresponding lifting $\rho^\infty \in \Lambda^\infty(\widehat{\mu^2})$ described in Section 1. If we assume that there exists a lifting $\upsilon \in \Lambda^\infty(\mu)$ satisfying (1) we get for $f := \chi_{[0,1/2)} - \chi_{(1/2,1]}$ that

$$1/2 = \rho_*(f^+)(1/2) \wedge \rho_*(f^-)(1/2) \le \upsilon(f^+)(1/2) \wedge \upsilon(f^-)(1/2) = 0,$$

a contradiction.

Nevertheless we will see below that (1) becomes true for some $\psi \in \mathscr{G}(\mu)$ instead of $\upsilon \in \Lambda^\infty(\mu)$ as a consequence of the theorem of Orlicz and Mazur which so far never seemed to have been applied in lifting theory.

In the next definition we introduce the *sublinear* and *superlinear lifting* by isolating those properties ρ_* and ρ^* listed in Proposition 2.1, which are essential for the application of the Mazur–Orlicz theorem.

DEFINITION 2.3. Given a measure space (Ω, Σ, μ) let us call a map from $\mathscr{L}^\infty(\mu)$ into itself with the properties (l1), (l2), and (l3) a *representation* for $\mathscr{L}^\infty(\mu)$. We call any representation with the additional properties (l4), (l6), and (l7) a *sublinear lifting*, and we call it a *superlinear lifting* if we replace (l7) by (l8). We write $\mathscr{B}(\mu)$ for the system of all sublinear liftings and $\mathscr{P}(\mu)$ for the system of all superlinear liftings for μ.

Note that any $\tau \in \mathscr{P}(\mu)$ is monotone; i.e., it satisfies condition (l5) (for τ instead of ρ_*).

Remark 2.4. It can be seen from the proof of Proposition 2.1 that the following two propositions hold true.

(i) From $\rho \in \mathscr{B}(\mu \hat{\otimes} \nu)$ follows $\rho^* \in \mathscr{B}(\mu)$.

(ii) From $\rho \in \mathscr{P}(\mu \hat{\otimes} \nu)$ follows $\rho_* \in \mathscr{P}(\mu)$.

Representations of the sort of Definition 2.3 appear naturally in a process given by von Neumann (see [10]) for passing from liftings for sets to liftings for functions, if this process is generalized in replacing liftings by densities in the following way. For $\delta \in \vartheta(\mu)$, $f \in \mathscr{L}^\infty(\mu)$, and $\omega \in \Omega$ define

$$\delta^0(f)(\omega) := \inf\{r \in \mathbf{Q} : \omega \in \delta(\{f < r\})\},$$

$$\delta_0(f)(\omega) := \sup\{r \in \mathbf{Q} : \omega \in \delta(\{f > r\})\}.$$

As they are not found explicitly in literature, most of the properties in the following proposition seem to be more or less folklore.

PROPOSITION 2.5. *We have $\delta_0 \in \mathscr{P}(\mu)$ and $\delta^0 \in \mathscr{B}(\mu)$ for $\delta \in \vartheta(\mu)$. In addition the following properties hold true for $f, g \in \mathscr{L}^\infty(\mu)$.*

(l12) $\delta_0(f + a) = \delta_0(f) + a$ *and* $\delta^0(f + a) = \delta^0(f) + a$ *for all* $a \in \mathbf{R}$,

(l13) $\delta_0(f \wedge g) = \delta_0(f) \wedge \delta_0(g)$,

(l14) $\delta^0(f \vee g) = \delta^0(f) \vee \delta^0(g)$,

(l15) $\delta_0(fg) \geq \delta_0(f)\delta_0(g)$,

and

(l16) $\delta^0(fg) \leq \delta^0(f)\delta^0(g)$.

It is well known that by the same definitions δ_0 and δ^0 can be extended to $\bar{\mathscr{L}}^0(\mu)$ and that under the usual conventions for addition, order, resp. multiplication similar properties hold true.

Remark 2.6. By Example 2.2 it can be seen that ρ_* resp. ρ^* does not enjoy the compatibilities (l13) resp. (l14) with lattice operations and this seems to make the difference between ρ_* resp. ρ^* on one side and δ_0 resp. δ^0 on the other as will be seen below. Indeed (keeping the notation of Example 2.2) we have $\rho_*(f^+ \wedge f^-)(1/2) = \rho_*(0)(1/2) = 0 < (\rho_*(f^+) \wedge \rho_*(f^-))(1/2) = 1/2$ resp. $\rho_*(f^+ \vee f^-)(1/2) = \rho_*(1)(1/2) = 1 > 1/2 = (\rho_*(f^+) \vee \rho_*(f^-))(1/2)$.

We will apply subsequently the following version of the Mazur–Orlicz theorem due to V. Ptak [11]:

(MOP) If X is a linear space and $\alpha, \beta: X \to \mathbf{R}$ are maps such that

1. $\alpha(x_1 + x_2) \leq \alpha(x_1) + \alpha(x_2)$ for $x_1, x_2 \in X$,

2. $\alpha(\lambda x) = \lambda\alpha(x)$ for $\lambda \geq 0$,

3. $\sum_{i=1}^{n} \lambda_i \beta(x_i) \leq \alpha(\sum_{i=1}^{n} \lambda_i x_i)$, for $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \geq 0$,

then there exists a linear map $f\colon X \to \mathbf{R}$ with

$$\beta(x) \leq f(x) \leq \alpha(x) \qquad \text{for } x \in X.$$

THEOREM 2.7. *If (Ω, Σ, μ) is a complete measure space, τ is a superlinear lifting for $\mathscr{L}^\infty(\mu)$, σ is a sublinear lifting for $\mathscr{L}^\infty(\mu)$, and $\tau \leq \sigma$ on $\mathscr{L}^\infty(\mu)$, then there exists a linear lifting φ for $\mathscr{L}^\infty(\mu)$ such that $\tau \leq \varphi \leq \sigma$.*

Proof. Clearly $\tau \leq \sigma$ on $\mathscr{L}^\infty(\mu)$. We apply (MOP) for $X := L^\infty(\mu)$ and

$$\alpha_\omega(\hat{f}) := \sigma(f)(\omega), \qquad \beta_\omega(\hat{f}) := \tau(f)(\omega)$$

if $f \in \hat{f} \in L^\infty(\mu)$ and $\omega \in \Omega$. Then α_ω clearly satisfies 1 and 2 of (MOP) for all $\omega \in \Omega$. If $\hat{f}_1, \dots, \hat{f}_n \in L^\infty(\mu)$ and $\lambda_1, \dots, \lambda_n \geq 0$ then

$$\sum_{i=1}^{n} \lambda_i \beta_\omega(\hat{f}_i) = \sum_{i=1}^{n} \beta_\omega(\lambda_i \hat{f}_i) \leq \beta_\omega\left(\sum_{i=1}^{n} \widehat{\lambda_i f_i}\right) \leq \alpha_\omega\left(\sum_{i=1}^{n} \widehat{\lambda_i f_i}\right);$$

i.e., 3 of (MOP) is satisfied.

Therefore for any $\omega \in \Omega$ there exists linear map $\varphi_\omega\colon L^\infty(\mu) \to \mathbf{R}$ with

$$\beta_\omega(\hat{f}) \leq \varphi_\omega(\hat{f}) \leq \alpha_\omega(\hat{f}) \qquad \text{for any } \hat{f} \in L^\infty(\mu).$$

Put

$$\varphi(f)(\omega) := \varphi_\omega(\hat{f}) \qquad \text{for any } f \in \mathscr{L}^\infty(\mu) \qquad \text{and} \qquad \omega \in \Omega.$$

It follows that

$$\tau(f) \leq \varphi(f) \leq \sigma(f) \qquad \text{for all } f \in \mathscr{L}^\infty(\mu)$$

and $\tau(f) = f = \sigma(f)$ a.e. (μ); hence $\varphi(f) \in \mathscr{L}^\infty(\mu)$ and $\varphi(f) = f$ a.e. (μ). Clearly $\varphi(f) = \varphi(g)$ if $f = g$ a.e. (μ) and φ is linear. For $0 \leq f \in \mathscr{L}^\infty(\mu)$ we have $\varphi(f) \geq \tau(f) \geq 0$, hence $\varphi \in \mathscr{G}(\mu)$. ∎

Remarks 2.8. (i) The completeness of the measure space, even if it is probability space, cannot be dispensed with in Theorem 2.7, since it is known from [1] that it is consistently true in Zermelo–Fraenkel set theory with axiom of choice (ZFC) to assume the non-existence of linear liftings in non-complete probability spaces.

(ii) It can be seen from the proof of Theorem 2.7 that condition (l4) is an essential property of superlinear resp. sublinear liftings for the conclusion of the theorem. If it would be dropped, representations with the remaining properties would exist in any measure space (Ω, Σ, μ), since there always exists a representation τ for $\mathscr{L}^\infty(\mu)$ which is a linear (not necessarily positive) map (see [12, proof of 1.2]). This would imply that in any complete measure space there exists a linear lifting φ, which is not true because in that case $\delta(A) := \{\varphi(\chi_A) \geq 1\}$ defines a density (see [6, p. 36]). But if $(A_i)_{i \in I}$ is a decomposition ND for (Ω, Σ, μ) then $(\delta(A_i))_{i \in I}$ is a decomposition D for (Ω, Σ, μ). But the latter decompositions do not exist generally. (See [2] for definitions and a counterexample.)

COROLLARY 2.9. *For any two complete probability spaces* (Ω, Σ, μ), (Θ, T, ν), *and a* $\rho \in \mathscr{G}(\mu \mathbin{\hat{\otimes}} \nu)$ *there exists a linear lifting* $\psi \in \mathscr{G}(\mu)$ *such that*

$$\rho_*(f) \leq \psi(f) \leq \rho^*(f) \qquad \textit{for any } f \in \mathscr{L}^\infty(\mu).$$

The proof is immediate from Proposition 2.1 and Theorem 2.7.

Remark 2.10. It follows from Remark 2.6 that in general the ψ appearing in Theorem 2.9 is no lattice operator and in particular no lifting. The example of Remark 2.6 is typical by the following reasoning.

If $\tau \in \mathscr{P}(\mu)$, $\rho \in \Lambda(\mu)$, and $\tau \leq \rho$ then $\tau(f) \wedge \tau(g) \leq \rho(f) \wedge \rho(g) = \rho(f \wedge g)$, the latter by [6, p. 35] for $f, g \in \mathscr{L}^\infty(\mu)$. Hence $\tau(f) \wedge \tau(g) = 0$ if $0 \leq f$, $g \in \mathscr{L}^\infty(\mu)$, and $f \wedge g = 0$ a.e. (μ).

Remark 2.11. If (Ω, Σ, μ) is a complete measure space then by Proposition 2.5 and Theorem 2.7 for any $\delta \in \vartheta(\mu)$ there exists a $\varphi \in \mathscr{G}(\mu)$ such that

$$\delta_0 \leq \varphi \leq \delta^0.$$

But there is a better result; i.e., there exists a $\rho \in \Lambda(\mu)$ such that

$$\delta_0 \leq \rho \leq \delta^0.$$

Indeed, by e.g., [13] there exists a $\rho \in \Lambda(\mu)$ such that

$$\delta(A) \subseteq \rho(A) \subseteq \delta^c(A) \qquad \text{for all } A \in \Sigma.$$

For all $f \in \mathscr{L}^\infty(\mu)$ and $r \in \mathbf{Q}$ we have (remember $\rho = \rho^0$) that $f\chi_{\{f > r\}} \geq r\chi_{\{f > r\}}$ implies $\rho(f)\chi_{\rho(\{f > r\})} \geq r\chi_{\rho(\{f > r\})}$. This implies $\rho(f)(\omega) \geq r$ for all $\omega \in \delta(\{f > r\}) \subseteq \rho(\{f > r\})$; hence $\rho(f)(\omega) \geq \delta_0(f)(\omega)$.

Starting from $f\chi_{\{f < r\}} < r\chi_{\{f < r\}}$ we find $\rho(f) \leq \delta^0(f)$.

Before the next result let us fix the following notation providing further simplifications. We can define for any representation φ for $\mathscr{L}^\infty(\mu)$ a new one by means of

$$\varphi^-(f) := -\varphi(-f) \qquad \text{for } f \in \mathscr{L}^\infty(\mu)$$

satisfying $(\varphi^-)^- = \varphi$. It follows that $\rho^* = (\rho_*)^-$ or $\rho_* = (\rho^*)^-$ and $(\delta_0)^- = \delta^0$ or $(\delta^0)^- = \delta_0$.

If $\delta^c(A) := [\delta(A^c)]^c$ for $A \in \Sigma$ and $\delta \in \vartheta(\mu)$ then δ^c is an upper density and $(\delta^c)^0 = \delta_0$ and $(\delta^c)_0 = \delta^0$. If $\rho \in \Lambda(\mu)$ then $\rho^c = \rho$ and $\rho^0 = \rho_0 = \rho$ if we do not distinguish between the liftings ρ for sets and the liftings $\bar{\rho}$ for functions in $\mathscr{L}^\infty(\mu)$ uniquely determined by $\bar{\rho}(\chi_A) = \chi_{\rho(A)}$ for $A \in \Sigma$.

We write $\mathscr{V}(\mu)$ resp. $\mathscr{W}(\mu)$ for the collection of all representations φ for $\mathscr{L}^\infty(\mu)$ which satisfy the property (l14) resp. (l13).

The following remarks are easy to verify.

Remarks 2.12. For any measure space (Ω, Σ, μ) we have

(i) $\varphi \in \mathscr{V}(\mu)$ if and only if $\varphi^- \in \mathscr{W}(\mu)$.

(ii) For any $\varphi \in \mathscr{P}(\mu)$ we have $\varphi^- \in \mathscr{B}(\mu)$ and $\varphi \leq \varphi^-$. The image of $\mathscr{P}(\mu)$ under the map $^- : \varphi \in \mathscr{P}(\mu) \to \mathscr{B}(\mu)$ is the class $\mathscr{B}^*(\mu)$ of all $\beta \in \mathscr{B}(\mu)$ with $\beta(f) \leq 0$ for $0 \geq f \in \mathscr{L}^\infty(\mu)$.

(iii) All $\beta \in \mathscr{V}(\mu) \cup \mathscr{W}(\mu) \cup \mathscr{P}(\mu) \cup \mathscr{B}^*(\mu) \cup \mathscr{G}(\mu)$ are monotone.

If we put a representation φ for $\mathscr{L}^\infty(\mu)$

$$\varphi_1(A) := \{\varphi(\chi_A) \geq 1\} \text{ resp. } \varphi^1(A) := \{\varphi(\chi_A) > 0\} \qquad \text{for } A \in \Sigma,$$

then φ_1 and φ^1 have the properties (l1) and (l2).

COROLLARY 2.13. *For any complete measure space (Ω, Σ, μ) the existence of a superlinear lifting φ implies the existence of a linear lifting ψ with $\varphi \leq \psi \leq \varphi^-$.*

Proof. For $\varphi \in \mathscr{P}(\mu)$ we have $\varphi^- \in \mathscr{B}(\mu)$ and $\varphi \leq \varphi^-$ by Remark 2.12. The assertion follows now from Theorem 2.7. ∎

LEMMA 2.14. *For any measure space (Ω, Σ, μ) and any representation φ for μ with $0 \leq \varphi(\chi_A) \leq 1$ for all $A \in \Sigma$ it follows that*

$$\chi_{\varphi_1(A)} \leq \varphi(\chi_A) \leq \chi_{\varphi^1(A)}$$

for $A \in \Sigma$.

Proof. For $\omega \in \varphi_1(A)$ follows $\chi_{\varphi_1(A)}(\omega) = 1 \leq \varphi(\chi_A)(\omega)$ but $0 = \chi_{\varphi_1(A)}(\omega) \leq \varphi(\chi_A)$ for all $\omega \notin \varphi_1(A)$ by assumption. For $\omega \in \varphi^1(A)$ we have $\chi_{\varphi^1(A)}(\omega) = 1 \geq \varphi(\chi_A)(\omega)$ but $0 = \chi_{\varphi^1(A)}(\omega) = \varphi(\chi_A)(\omega)$ for all $\omega \notin \varphi^1(A)$ by assumption. ∎

PROPOSITION 2.15. *Let a representation φ for $\mathscr{L}^\infty(\mu)$ be given. If φ satisfies* (l7) *resp.* (l8) *then it follows that $(\varphi^1)^c = (\varphi^-)_1$ resp. $(\varphi_1)^c = (\varphi^-)^1$.*

Proof. Throughout let $A \in \Sigma$, $\omega \in \Omega$ and note first

(1) $(\varphi_1)^c(A) = \{\varphi(\chi_{A^c}) < 1\}$,

(2) $(\varphi^1)^c(A) = \{\varphi(\chi_{A^c}) \le 0\}$,

(3) $(\varphi^-)_1(A) = \{\varphi(-\chi_A) \le -1\}$,

and

(4) $(\varphi^-)^1(A) = \{\varphi(-\chi_A) < 0\}$.

If φ satisfies (17) then

(5) $\varphi(\chi_{A^c})(\omega) = \varphi(1 - \chi_A)(\omega) \le 1 + \varphi(-\chi_A)(\omega)$

and this implies together with (2) and (3)

(6) $(\varphi^-)_1 \le (\varphi^1)^c$.

If φ satisfies (18) then

(7) $\varphi(\chi_{A^c})(\omega) = \varphi(1 - \chi_A)(\omega) \ge 1 + \varphi(-\chi_A)(\omega)$

and this implies together with (1) and (4)

(8) $(\varphi^-)^1 \ge (\varphi_1)^c$.

Next note that $\varphi(1) = 1$ is equivalent with $\varphi(-1) = -1$. So if φ satisfies (18) then φ^- satisfies (17) and (6) implies $((\varphi^-)^-)_1 \le ((\varphi^-)^1)^c$; i.e. $\varphi_1 \le ((\varphi^-)^1)^c$, i.e., $(\varphi_1)^c \ge (\varphi^-)^1$ which gives together with (8) that $(\varphi_1)^c = (\varphi^-)^1$.

If φ satisfies (17) then φ^- satisfies (18) and (8) implies $((\varphi^-)^-)^1 \ge ((\varphi^-)^1)^c$; i.e., $\varphi^1 \ge ((\varphi^-)_1)^c$, i.e., $(\varphi^1)^c \le (\varphi^-)_1$ which gives together with (6) that $(\varphi^1)^c = (\varphi^-)_1$. ∎

PROPOSITION 2.16. *Let a monotone representation φ for $\mathscr{L}^\infty(\mu)$ be given. If φ satisfies* (17) *then it follows that $\varphi^1 \in Y(\mu)$.*

Proof. For $A, B \in \Sigma$ we have

$$\varphi(\chi_{A \cup B}) \le \varphi(\chi_A + \chi_B) \le \varphi(\chi_A) + \varphi(\chi_B).$$

Consequently $\varphi^1(A \cup B) \subseteq \varphi^1(A) \cup \varphi^1(B)$. But the converse inclusion holds true since φ^1 is monotone which follows from the monotonicity of φ.

$\varphi(1) = 1$ implies $\varphi^1(\Omega) = \Omega$ while $\varphi(0) = 0$ implies $\varphi^1(\emptyset) = \emptyset$. Hence $\varphi^1 \in Y(\mu)$. ∎

It follows from the proof of Proposition 2.15 that both Propositions 2.15 and 2.16 hold true under the weaker assumption "$\varphi(a) = a$ for $a = 0, 1$" instead of the property (13) for φ.

PROPOSITION 2.17. *Let a monotone representation φ for μ be given. If φ satisfies* (18) *then it follows that $\varphi_1 \in \vartheta(\mu)$.*

Proof. First note that φ^- is monotone if φ is and that $\varphi^-(a) = a$ for $a = 0, 1$. φ^- satisfies (l7) since φ satisfies (l8). Hence $(\varphi^-)^1 \in Y(\mu)$ by Proposition 2.16 and from Proposition 2.15 we get $\varphi_1 = ((\varphi^-)^1)^c \in \vartheta(\mu)$. ∎

It follows from the above proof that Proposition 2.17 remains true under the weaker assumption "$\varphi(a) = a$ for $a = -1, 0, 1$" instead of (l8) for φ.

PROPOSITION 2.18. *Let $\varphi \in \mathscr{P}(\mu)$ be given, satisfying condition* (l11). *Then it follows that $\varphi_1 \in \vartheta(\mu)$ and $(\varphi_1)_0 \leq \varphi$. In particular this conclusion holds true for all $\varphi \in \mathscr{G}(\mu)$.*

Proof. First note that φ is positive homogeneous and monotone; in particular $0 \leq \varphi(\chi_A) \leq 1$ for all $A \in \Sigma$. By Proposition 2.17 we have $\varphi_1 \in \vartheta(\mu)$ and hence $(\varphi_1)_0 \in \mathscr{P}(\mu)$ by Proposition 2.5.

(1) Let us assume first that $0 \leq f \in \mathscr{L}^\infty(\mu)$ and let be $0 \leq r \in \mathbf{Q}$. Then $f \geq f\chi_{\{f > r\}} \geq r\chi_{\{f > r\}}$ which implies by monotonicity, positive homogeneity, and Lemma 2.14 that $\varphi(f) \geq r\chi_{\varphi_1(\{f > r\})}$. This gives by definition of $(\varphi_1)_0$ immediately that $(\varphi_1)_0(f) \leq \varphi(f)$.

(2) For arbitrary $f \in \mathscr{L}^\infty(\mu)$ note that $0 \leq f + \|f\|_\infty \in \mathscr{L}^\infty(\mu)$. Hence by (1) and since (l11) is true for φ and for $(\varphi_1)_0$ it follows that $(\varphi_1)_0(f) + \|f\|_\infty = (\varphi_1)_0(f + \|f\|_\infty) \leq \varphi(f + \|f\|_\infty) = \varphi(f) + \|f\|_\infty$. Therefore $(\varphi_1)_0 \leq \varphi$. ∎

Remark 2.19. Let ρ, ψ be given as in Theorem 2.9 and write $\varphi := \rho_*$ for simplicity. Then it follows that $\rho = \varphi^-$; hence $\varphi \leq \psi \leq \varphi^-$ with $\psi \in \mathscr{G}(\mu)$. This implies by [6, p. 37] that $\varphi_1 \leq \psi_1 \leq \psi^1 = \psi_1^c$ and $\psi_1 \in \vartheta(\mu)$, $\psi^1 \in Y(\mu)$. By Proposition 2.11 there exists a $\lambda \in \Lambda(\mu)$ with $(\psi_1)_0 \leq \lambda \leq (\psi_1)^0$. This implies that

$$(\varphi_1)_0 \leq (\psi_1)_0 \leq \lambda \leq (\psi_1)^0 \leq (\varphi_1)^0.$$

But Example 2.2 implies that we do not have $\varphi \leq (\varphi_1)_0$ or $\varphi \leq (\psi_1)_0$ generally. To the contrary $(\varphi_1)_0 \leq \varphi$ by Proposition 2.18 as well as $(\varphi_1)_0 \leq (\psi_1)_0 \leq \psi \leq \varphi^-$. In particular $(\varphi_1)_0 \neq \varphi$ and clearly $(\psi_1)_0 \neq \psi$ for all $\psi \in \mathscr{G}(\mu) \setminus \Lambda(\mu)$. Otherwise for a $\lambda \in \Lambda(\mu)$ we have $\psi = (\psi_1)_0 \leq \lambda \leq \psi^0 = ((\psi_1)_0)^- = \psi^- = \psi$; hence $\psi = \lambda$, a contradiction.

The next theorem is another consequence of the main Theorem 2.7.

THEOREM 2.20. *If (Ω, Σ, μ) is a complete measure space then*

$$\mathscr{P}_{\mathrm{maxl}}(\mu) = \mathscr{B}^*_{\mathrm{minl}}(\mu) = \mathscr{G}(\mu),$$

where $\mathscr{P}_{\mathrm{maxl}}(\mu)$ resp. $\mathscr{B}^*_{\mathrm{minl}}(\mu)$ denotes the system of all maximal resp. minimal elements of $\mathscr{P}(\mu)$ resp. $\mathscr{B}^*(\mu)$.

Proof. (1) If $\varphi \in \mathscr{P}_{\text{maxl}}(\mu)$ then by Theorem 2.7 choose a $\psi \in \mathscr{G}(\mu)$ with $\varphi \leq \psi \leq \varphi^{-}$. Since $\psi \in \mathscr{P}(\mu)$ it follows that $\varphi = \psi \in \mathscr{G}(\mu)$.

(2) If conversely $\psi \in \mathscr{G}(\mu)$, $\varphi \in \mathscr{P}(\mu)$, and $\psi \leq \varphi$ then with Remark 2.12, (ii) follows $\varphi \leq \varphi^{-} \leq \psi^{-} = \psi$;a hence $\varphi = \psi$, i.e., $\psi \in \mathscr{P}_{\text{maxl}}(\mu)$.

(3) Now apply the order anti-isomorphism $\varphi \to \varphi^{-}$ for getting $\mathscr{B}^{*}_{\text{minl}}(\mu) = \mathscr{G}(\mu)$. ∎

PROPOSITION 2.21. *For any complete measure space* (Ω, Σ, μ) *the spaces* $(\mathscr{P}(\mu), \leq)$ *and* $(\mathscr{B}^{*}(\mu), \geq)$ *are inductively ordered.*

Proof. For a given chain $\mathscr{C}$ in $\mathscr{P}(\mu)$ define $\varphi(f) := \sup_{\gamma \in \mathscr{C}} \gamma(f)$ for all $f \in \mathscr{L}^{\infty}(\mu)$. Let $f, g \in \mathscr{L}^{\infty}(\mu)$ and note that for all $\gamma_1, \gamma_2 \in \mathscr{C}$ there exists a $\gamma \in \mathscr{C}$ with $\gamma_1, \gamma_2 \leq \gamma$. Then $\varphi(f+g) \geq \gamma(f+g) \geq \gamma(f) + \gamma(g) \geq \gamma_1(f) + \gamma_2(g)$; hence $\varphi(f+g) \geq \gamma_1(f) + \gamma_2(g)$ for all $\gamma_1, \gamma_2 \in \mathscr{C}$. This implies that $\varphi(f+g) \geq \varphi(f) + \gamma_2(g)$ for all $\gamma_2 \in \mathscr{C}$ which implies (l8) for φ. But φ clearly satisfies (l2), (l3), (l4), and (l6). since for some $\gamma \in \mathscr{C}$ we have $\gamma \leq \varphi$ it follows that $\gamma(f) \leq \varphi(f) \leq \varphi^{-}(f) \leq \gamma(f)^{-}$ for all $f \in \mathscr{L}^{\infty}(\mu)$. As (Ω, Σ, μ) is complete this implies that $\varphi(f) \in \mathscr{L}^{\infty}(\mu)$ and $\varphi(f) = f$ a.e. (μ); i.e., (l1) holds true for φ. Consequently $\varphi \in \mathscr{P}(\mu)$. ∎

Remark 2.22. For complete measure spaces, (Ω, Σ, μ) follows by the Zorn–Kuratowski lemma from the last two results $\mathscr{G}(\mu) \neq \emptyset$ if $\mathscr{P}(\mu) \neq \emptyset$. But $\mathscr{P}(\mu) \neq \emptyset$ can be inferred from $\vartheta(\mu) \neq \emptyset$ by Proposition 2.5. By [1] this implies that in Proposition 2.21 the completeness of the basic measure space cannot be dropped since for arbitrary totally finite measure spaces densities exist according to [4].

Remark 2.23. If we denote by $\mathscr{Q}(\mu)$ the system of all $\varphi \in \mathscr{P}(\mu)$ which satisfy (l13) in addition we get $\Lambda^{\infty}(\mu) \subseteq \mathscr{Q}_{\text{maxl}}(\mu)$ for arbitrary measure spaces, but it is an open problem whether the converse inclusion holds true. It can be seen in the same way as in the proof of Proposition 2.21 that for complete measure spaces the space $(\mathscr{Q}(\mu), \leq)$ is inductively ordered. And again by Proposition 2.5 $\mathscr{Q}(\mu) \neq \emptyset$ if $\vartheta(\mu) \neq \emptyset$, so that in the latter case the Zorn–Kuratowski lemma implies $\mathscr{Q}_{\text{maxl}}(\mu) \neq \emptyset$. If we replace $\mathscr{Q}(\mu)$ by the smaller set $\mathscr{P}_0(\mu) := \{\delta_0 : \delta \in \vartheta(\mu)\}$ in the above open problem then the next result gives a solution.

For the following we should carefully distinguish between $\Lambda(\mu)$ and $\Lambda^{\infty}(\mu)$ (see Preliminaries). Note that for $\rho \in \Lambda(\mu)$ clearly $\rho_0 = \rho^0 = \rho^{\infty}$ and for $\rho^{\infty} \in \Lambda^{\infty}(\mu)$ always $(\rho^{\infty})_1 = (\rho^{\infty})^1 = \rho$.

Remark 2.24. For complete measure space (Ω, Σ, μ) we have

(i) $(\mathscr{P}_0)_{\text{maxl}}(\mu) = \Lambda^{\infty}(\mu)$

and

(ii) $\vartheta_{\mathrm{maxl}}(\mu) = Y_{\mathrm{minl}}(\mu) = \Lambda(\mu)$.

For (i) note that $\Lambda^{\infty}(\mu) \subseteq \mathscr{Q}_{\mathrm{maxl}}(\mu)$ by Remark 2.22; consequently $\rho^{\infty} \in (\mathscr{P}_0)_{\mathrm{maxl}}(\mu)$ for any $\rho^{\infty} \in \Lambda^{\infty}(\mu)$ since $\mathscr{P}_0(\mu) \subseteq \mathscr{Q}(\mu)$. On the other hand, if $\delta_0 \in (\mathscr{P}_0)_{\mathrm{maxl}}(\mu)$, then choose $\rho \in \Lambda(\mu)$ with $\delta \leq \rho$ by [13]. It follows that $\delta_0 \leq \rho_0 \in \mathscr{P}_0(\mu)$; hence $\delta_0 = \rho_0 = \rho^{\infty} \in \Lambda^{\infty}(\mu)$ by maximality of δ_0. Remark (ii) follows in a similar way.

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